with a publication by the philosopher Hegel, in which he severely criticised astronomers for not paying more attention to philosophy, a science, said he, which would have shown them at once that there could not possibly be more than seven planets, and a study of which would have prevented, therefore, an absurd waste of time in looking for what in the nature of things could not be found. This is only one instance of the many refutations of dogmatic statements of philosophers who presage nature's laws without confirming them by actual observations.

However, the new planet was seen under conditions so unfavorable as to render it almost impossible to forecast its orbit. Fortunately the observations of the planet were communicated to Gauss. Gauss made use of the fact that six quantities known as elements completely determine the motion of a planet unaffected by perturbations. Since each observation of a planet gives two of these, e. g., the right ascension and declination, therefore three observations are sufficient to determine the six quantities and therefore to completely determine the planet's motion. Gauss applied this method and that of least squares and his his analysis proved a complete success,* the planet being rediscovered at the end of the year in nearly the position indicated by his calculations. This success proved him to be the greatest of astronomers as well as the greatest of "arithmeticians."

The attention excited by these investigations procured for him in 1807 the offer, from the Emporer of Russia, of a chair in the Academy ot St. Petersburg. But Gauss, having a marked objection to a mathematical chair, by the advice of the astronomer Olbers, who desired to secure him as director of a proposed new observatory at Göttingen, declined the offer of the emperor and accepted the position at Göttingen. He preferred this position because it afforded him an opportunity to devote all his time to science. He spent his life in Göttingen in the midst of continuous work and after his appointment never slept away from his observatory except on one occasion when he accepted an invitation from Humbolt† and attended a scientific congress at Berlin, in 1828. The only other time that he was absent from Göttingen was in 1854, when a railroad was opened between Göttingen and Hanover.‡

For some years after 1807 his time was almost wholly occupied by work connected with his observatory. In 1809 he published at Hamburg his *Theoria Motus Corporum Coelestium*, a treatise which contributed largely to the improvement of practical astronomy, and introduced the principle of curvilinier triangulation. In this treatise are found four formulæ in spherical trigonometry, commonly called "Gauss's Analogies," but which were published somewhat earlier by Karl Brandon Mollweide of Liepzig, 1774-1825, and still earlier by Jean Baptiste Joseph Delambre (1749-1822).‖ On observations in general (1812-1826) we have his memoir. *Theoria Combinationis Observationum Erroribus Minimis Obnoxia*, with a second part and supplement.

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*Berry's *A Short History of Astronomy.*
†*Britannica Encyclopedia*, 9th edition, Vol. X, page 104.
‡Cajori's *A History of Mathematics.*
‖Cajori's *A History of Mathematics.*

A little later he took up the subject of geodesy and from 1821 to 1848 acted as scientific adviser to the Danish and Hanoverian governments for the survey then in progress. His papers of 1843 and 1866, *Ueber Gegenstände der höhern Geodäsie*, contain his researches on the subject.

Gauss's researches on *Electricity and Magnetism* date from about the year 1830. In 1833 he published his first memoir on the theory of magnetism, the title of which is *Intensitas vis Magneticæ Terrestris ad Mensuram Absolutam Revocata*. A few months afterward he, together with Weber, invented the declination instrument and bifilar magnetometer. The same year they erected at Göttingen a magnetic observatory free from iron (as Humbolt and Arago had previously done on a smaller scale), where they made magnetic observations and showed in particular that it was possible and practical to send telegraphic signals, having sent telegraphic signals to neighboring towns. At this observatory he founded an association called the *Magnetische Varein*, composed at first almost entirely of Germans, whose continuous observations at fixed times extended from Holland to Sicily. The volumes of their publications, *Resultate aus der Beobachtungen des Magnetischen Vereins*, extend from 1833 to 1839. In these volumes for 1838 and 1839 are contained two important memoirs by Gauss, one on the general theory of earth-magnetism, the other on the theory of forces attracting according to the inverse squares of the distance. Like Poisson, he treated the phenomena in electrostatics as due to attractions and repulsions between imponderable particles. In electro-dynamics he arrived, in 1835, at a result equivalent to that given by W. E. Weber in 1846, viz: that the attraction between two electrified particles, e and e', whose distance apart is r, depends on their relative motion and position according to the formula

$$ee' r^{-2}\{1+(rd^2r-\tfrac{1}{2}dr^2)^2c^{-2}\}.$$

Gauss, however, held that no hypothesis was satisfactory which rested on a formula and was not a consequence of physical conjecture, and as he could not form a plausible physical conjecture he abandoned the subject. Such conjectures were proposed by Riemann in 1858, and by C. Neumann and E. Betti in 1868, but Helmholtz in 1870, 1873 and 1874 showed that these conjectures were untenable.

In 1833, in a memoir on capillary attraction, he solved a problem in the Calculus of Variation, involving the variation of a certain double integral, the limits of integration also being variable; it is the earliest example of the solution of such a problem.

In 1846 was published his *Dioptrische Untersuchungen*, researches on optics, including systems of lenses.

As has already been observed, Gauss's most celebrated work in pure mathematics is the *Disquisitiones Arithmeticæ*, and a new epoch in the theory of numbers dates from the time of its publication. This treatise, Legendre's *Théorie des nombres* and Derichlet's *Vorlesunger über Zahlentheorie* are the standards on the Number Theory.

In this work Gauss has discussed the solution of binominal equations of the form $x^n=1$, which involves the celebrated theorem that the only regular polygons which can be constructed by elementary geometry are those of which the number of sides is $2^m(2^n+1)$, where m and n are integers and 2^n+1 is a prime. These equations are called *cyclotomic equations*, when n is prime and when they are satisfied by a primitive nth root of unity.

Gauss developed the theory of ternary quadratic forms involving two indeterminates, and also investigated the theory of determinants on whose results Jacobi based his researches on this subject.

The theory of Functions of Double Periodicity had its origin in the discoveries of Abel and Jacobi. Both arrived at the Theta Functions which play so large a part in the Theory of Double Periodic Functions. But Gauss had independently and at a far earlier date discovered these functions and their chief properties, having been led to them by certain integrals which occurred in the *Determinatio Attractionis*, to evaluate which he invented the transformation now associated with the name of Jacobi. In the memoir, *Determinatio Attractionis*, it is shown that the secular variations, which the elements of the orbit of a planet experience from the attraction of another planet which disturbs it, are the same as if the mass of the disturbing planet were distributed over its orbit into an elleptic ring in such a manner that equal masses of the ring would correspond to arcs of the orbit described in equal times.

Gauss's collected works have been published by the Royal Society of Göttingen, in seven 4-to volumes, 1863-1871, under the editorship of E. J. Schering. They are as follows: (1) The *Disquisitiones Arithmeticæ*, (2) *Theory of Numbers*, (3) *Analysis*, (4) *Geometry and Method of Least Squares*, (5) *Mathematical Physics*, (6) *Astronomy*, and (7) *Theoria Motus Corporium Cœlestium*. These include besides his various works and memoirs, notices by him of many of these, and of works of other authors in the *Göttingen gelehrte Anzeigen*, and a considerable amount of previously unpublished matter, *Nachlass*. Of the memoirs in pure mathematics, comprised for the most part in volumes ii, iii and iv (but to these must be added those on *Attraction* in volume v), there is not one which has not signally contributed to the branch of mathematics to which it belongs, or which would not require to be carefully analyzed in a history of the subject.

His collected works show that this wonderful mind had touched hidden laws in Mathematics, Physics and Astronomy, and every one of the subjects which he investigated was greatly extended and enriched thereby. He was also well versed in general literature and the chief languages of modern Europe, and was a member of nearly all the leading scientific societies in Europe.

He was the last of the great mathematicians whose interests were nearly universal. Since his time, the literature of most branches of mathematics has grown so rapidly that mathematicians have been forced to specialize in some particular department or departments.

Gauss was a contemporary of Lagrange and Laplace, and these three, of which he was the youngest, were the great masters of modern Analysis. In

Gauss that abundant fertility of invention which was marvelously displayed by the mathematicians of the preceding period, is combined with an absolute rigorousness in demonstration which is too often wanting in their writings. Lagrange was almost faultless both in form and matter, he was careful to explain his procedure, and, though his arguments are general, they are easy to follow. Laplace, on the other hand, explained nothing, was absolutely indifferent to style, and, if satisfied that his results were correct, was content to leave them either without a proof or even a faulty one. Many long and abstruse arguments were passed by with the remark, "it is obvious." This led Dr. Bowditch, of Harvard university, while translating Laplace's *Mechanique Céleste*, to say that whenever he came to Laplace's "it is obvious," he expected to put in about three weeks of hard work in order to see the obviousness. Gauss, in his writings, was as exact and elegant as Lagrange, but even more difficult to follow than Laplace, for he removed every trace of the analysis by which he reached his results, and even studied to give a proof which, while rigorous, should be as concise and synthetical as possible. He said: "Mathematics is the queen of the sciences, and arithmetic is the queen of mathematics," and his *Disquisitiones* confirms the statement.

Gauss had a strong will, and his character showed a curious mixture of self-conscious dignity and child-like simplicity. He was little communicative, and at times morose.

He possessed a remarkable power of attention and concentration, and in this power lies the secret of his wonderful achievements. As a proof of this power of attention we quote from Carpenter's *Mental Physiology*. Gauss, while engaged in one of his most profound investigations, was interrupted by a servant who told him that his wife (to whom he was known to be deeply attached, and who was suffering from a severe illness) was worse. "He seemed to *hear* what was said, but either did not comprehend it or immediately forgot it, and went on with his work. After some little time, the servant came again to say that his mistress was much worse, and to beg that he would come to her at once; to which he replied: 'I will come presently.' Again he lapsed into his previous train of thought, entirely forgetting the intention he had expressed, most probably without having distinctly realized to himself the import either of the communication or of his answer to it. For not long afterwards when the servant came again and assured him that his mistress was dying and that if he did not come immediately he would probably not find her alive, he lifted up his head and calmly replied, 'Tell her to wait until I come,'—a message he had doubtless often before sent when pressed by his wife's request for his presence while he was similarly engaged."

In bringing this imperfect sketch to a close, we wish to call attention to the fact that it has been conclusively shown that Gauss was not the first to give a satisfactory representation of complex numbers in a plane, this having been first satisfactorily done by Casper Wessel in 1797, though Wallis had made some use of graphic representation of complex numbers as early as 1785. Gauss

needs no undue credit to make him famous—the writing alone of any one of the seven of his collected works being sufficient to rank him among the great mathematicians of his day. However, it was Gauss who in 1831, "by means of his great reputation, made the representation of imaginery quantities in the 'Gaussian plane' the common property of all mathematicians." He brought also into general use the sign i for $\sqrt{-1}$, though it was first suggested by Euler. He called $a+bi$ a *complex number* and called a^2+b^2 the *norm*.

THE POPULARIZATION OF NON-EUCLIDEAN GEOMETRY.

By GEORGE BRUCE HALSTED.

In a charming article in the *Popular Science Monthly* for January, 1901, entitled, "Geometry: Ancient and Modern," Edwin S. Crawley delightfully helps the cultured reader to get his orientation in this subject for a start into the new century.

But strangely enough this admirable paper becomes somewhat obscure when it becomes tri-dimensional. It says: "If we proceed beyond the domain of two-dimensional geometry we merge the ideas of non-Euclidean and hyper-space."

If we do so, we are apt to blunder. Just as the Bolyai plane is utterly independent of the Euclidean plane, so the triply extended space of Bolyai is utterly independent of any Euclidean space or hyper-space.

The idea that tri-dimensional Bolyai space needs four-dimensional Euclidean space is an error into which many philosophers and some mathematicians have been led, perhaps from the unfortunate adoption of the name "radius of space curvature" for the space-constant.

This blunder was refuted even before it was born by Bolyai's geodesic geometry of limit surfaces.

Thereby a Euclidean plane can be represented by a surface in Bolyai space, the theorems of Euclidean geometry find their realization as surface theorems in non-Euclidean space, where the geodesic geometry is that of Euclidean straight lines in a Euclidean plane.

Because this error about "curvature in space" is so widespread and so insidious, I treated it fully in my "Report on Progress in Non-Euclidean Geometry" to the American Association for the Advancement of Science.

The very next sentence in Professor Crawley's article reads as follows: "The ordinary triply-extended space of our experience is purely Euclidean."

Here our author states not only something which is not known, but, strangely enough, something which never can be known, which never can be proven.

Man's metric knowledge of the world independent of man, coming through imperfect instruments, for example the eye, cannot be absolute and exact.

The pure idea of a perfect plane is a creation of the human mind.

When the variations in the approximately plane surface of an actual body are minute, we deliberately make the perceived imperfections disappear, that we may identify the surface we seem to see with our ideal creation, the perfect plane. Surface is an ideal or imaginery creation to which we fit even the apparent (not real) boundaries of physical objects.

Just so the straight line is a non-real creation.

In the theoretical, the scientific, the mathematical handling of any empirical data the process is always the same. Always the results of any observations hold good only within definite limits of exactitude and under particular conditions; we replace these results with statements of absolute precision and generality.

Our replacement is only confined in its free arbitrariness in that it should seem to snuggle to the seeming facts, and must introduce no logical contradictions.

In this sense the ordinary triply-extended space of our experience is at present Euclidean or Bolyaian or Riemannian as you choose. Each is, up to the present day, in simple and perfect harmony with experience, with experiment, with the properties of the solid bodies and the motions about us.

If the angle-sum of a single rectilineal triangle is exactly a straight angle, space is Euclidean; if less than a straight angle, Bolyaian; if more, Riemannian.

The mechanics of actual bodies in the external space of our experience might conceivably be shown by merely approximate measurements (the only kind that ever were) to be non-Euclidean; just as a body might be shown to weigh more than two grams or less than two grams, though it never can be shown to weigh precisely, absolutely two grams.

In this sense the Euclidean geometry is positively hopeless, in that it never can be proven and no respectable person would now for a moment attempt to establish it, while there was nothing theoretically absurd in the claim of C. S. Peirce to be able to show that space is Bolyaian, which claim has found its way into Boyer's *Histoire des Mathématiques*, page 247, though the index confuses C. S. Peirce with his father, Benjamin Peirce.

When Lobachevski exerted himself to obtain with exceeding great precision the sum of the three angles in the very largest triangles attainable for his measurement, he found this sum did not differ from two right angles by the hundredth part of a second. This shows that the space of experience approaches the ideal Euclidean space with an approximation which is very far-reaching. But it would be strange if any educated person should need to be told that all our measurements are approximate only, and that with the approximate we can never reach the absolutely exact.

The mistake of supposing that they *know* our space, "the space in which we really live," is not Boylyaian, made by Phillips and Fisher, Professors in

Yale, in their Elements of Geometry, page 23, and even by so good a mathematician as H. Schubert of Hamburg, I have exposed in my paper, "Non-Euclidean Geometry," AMERICAN MATHEMATICAL MONTHLY, Vol. 7, pages 123-133.

A striking testimony that the non-Euclidean geometry has won its fight and henceforth must be reckoned with is found in the History of Mathematics by Boyer, Paris, 1900. He says, pages 240-7: "The last quarter of the nineteenth century has seen built up interesting theories. But beyond contradiction the most original researches of this period pertain to the *non-Euclidean Geometries*, and it is with them that we will terminate this exposé of contemporary science."

I only wish the short accounnt which follows this prelude and terminates the book were as accurate as it is impressive and stimulating. A full-page portrait of Lobachevski is given. But of the two entirely different likenesses we possess of the great Russian, this is the conventional one of Lobachevski depressed, baffled, about to become blind and die with his great achievement unrecognized. The other portrait, a Daguerreotype from life, which I first saw at Kazan, and of which now you may see a copy as frontispiece of Engel's magnificent "Nikolaj Iwanowitsch Lobatschefskij," is a picture of Lobachevski the fighter, the dare-devil, the irrepressible, who startled and scandalized the despotic authorities of Kazan and the University by shooting off his rocket, who contemptuously overthrew the great Legendre, of Lobachevski who knew he was right against a scornful world, who has given to us a new freedom to explain and understand our universe and ourselves.

Boyer goes on as follows: "From far in the past men have striven to demonstrate the famous axiom postulated twenty centuries ago by Euclid, to-wit: Through a point can be drawn only one parallel to a given straight. These attempts remained unfruitful. However, at the end of the eighteenth century, an Italian jesuit, Saccheri, attempted to found a geometry resting on a principle different from the celebrated postulate." This is a mistake. Saccheri's was simply one more attempt to prove the postulate, and he thought he had proven it. The title of his book is: *Euclides ab omni naevo vindicatus.* THE AMERICAN MATHEMATICAL MONTHLY has the honor of being the first to publish a translation of this now famous work into any modern language.

Boyer continues: "Finally at the beginning of the nineteenth century, a Russian, Lobachevski, and a Hungarian, John Bolyai, perceived at about the same time the impossibility of this demonstration. Their works published independently one of the other had without doubt been inspired by the doctrines of the philosopher Kant, who, in a passage of his *Kritik* of pure reason, indicated a new consideration of space. For this latter, space existed *a priori*, preceding all experience, as form completely subjective of our intuition."

The statement that the work of John Bolyai was inspired by Kant is a complete mistake. It is a gratuitous assumption cut out of whole cloth. There is nothing to show that John Bolyai ever even heard of the existence of Kant. At Maros-Vásárhely I examined the papers, the correspondence, the

Nachlass of John Bolyai. There was not the slightest mention of Kant, not the remotest reference to Kant.

Lobachevski knew of Kant through the professor of physics at Kazan, Bronner, once an admirer of the *Kritik der reinen Vernunft*. But Lobachevski tells us more than once the inspiration and mental ancestry of his achievement, and there is no place for Kant.

Kant, supposing that we knew Euclid's geometry and Aristotle's logic to be true absolutely and necessarily, accounted for the paradox by teaching that this seemingly universal synthetic knowledge was in reality particular, being part of the apparatus of the human mind itself. When Boole and De Morgan made new kinds of logic of which the Aristotelian is only a special case, when Lobachevski, Bolyai and Riemann made new kinds of geometry of which the Euclidean is only a special case, then the very foundations were cut away from under the Kantian system of philosophy, and it was left like a man trying to lift himself by his own boots.

The Scotch philosophy accounts for this supposed absolute, metrically exact knowledge, by teaching that there are in the human mind certain synthetic theorems, called by them intuitions, directly God-given. Dr. McCosh summed up this Scotch doctrine in a big book entitled, "The Intuitions of the Mind, Inductively Investigated."

One of these divinely implanted intuitions was Euclid's famous parallel-postulate!! *Voila!*

"Yet," to quote a sentence from Wenley's criticism in Science of McCosh's disciple Ormond, "we may well doubt whether a thinker standing with one foot firmly planted on the Rock of Ages, and the other pointing heavenward, has struck the attitude most conducive to progress."

After a brief biography of Lobachevski and a quotation from his Introduction to New Principles of Geometry, which Introduction I have translated into English and published as volume V of the Neomonic Series, Boyer continues: "This postulated, behold how he proceeds to the development of his doctrine. He announces at the beginning the following axiom: Through a point can be drawn *many parallels* to a given straight." To say the least this is likely to produce misconception.

Lobachevski assumes that through a given point outside a given straight can be drawn many straights which never meet the given straight, but of this sheaf of non-meeters only two are parallel to the given straight, namely, the two which approach the given straight asymptotically.

Again Boyer says, page 245: "He became professor, and when he died in 1856 he occupied the position of Rector of the University where he had entered as simple student." This is another mistake. When Lobachevski died in 1856 he had been displaced from his position as Rector for ten years.

Passing on to the Riemannian geometry, Boyer says: "To construct this Geometry, its inventor rejects the postulatum and the first axiom of Euclid: two points determine a straight." In fact neither of these assumptions of Euclid

need be rejected to get a Riemannian geometry, the "Single Elliptic Geometry."

Boyer has perhaps been misled by his own paraphrases for what really occurs in Euclid. The real postulatum is as follows: "And if a straight cutting two straights makes with them angles interior and lieing on the same side, which together are less than two right angles, then the two straights indefinitely produced cut each other on the side on which these angles lie."

Needless to say, this remains true in both single and double elliptic geometry. The postulate currently taken in place of the unwieldy postulatum of Euclid is "that two straight lines, which cut one another, cannot be both parallel to the same straight line," which is credited by Playfair to Ludlam, though attributed even by Cajori to Playfair, and currently called Playfair's axiom. Even this does not help toward Riemannian spaces, for in them there are no parallels in the sense of coplanar non-meeting straights.

Finally Boyer gives us the well-known confusion in connection with Beltrami's pseudo-sphere.

John Bolyai found a surface in Lobachevskian space whose geodesic geometry is that of Euclid's straights. No one supposed that this reduced Euclidean to be only a branch of Bolyaian geometry.

Beltrami found a surface in Euclidean space whose geodesic geometry is that of Bolyai's straights. Boyer says, page 246, this reduced the Geometry of Lobachevski to be only a branch of ordinary geometry. On the contrary, the truth is that Euclidean geometry is only that special case of Bolyaian geometry made by assuming the space-constant as infinite.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

134. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

A certain piece of land is surrounded by a four-board fence, the boards being 16 feet long. The number of acres in the land equals the number of boards in the fence. How many acres in the land?

I. Solution by G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.; P. S. BERG, B. Sc., Larimore, N. D.; and MARTIN SPINKS, Wilmington, O.

(1) *When the field is in the form of a square.*

Let $ABCD$ be the square, O its center, $FG=l=16$ feet=the length of a panel of the fence, and $n=4$=the number of boards in a panel. Then the area of the triangle $OFG=\frac{1}{2}OE\times FG=\frac{1}{2}OE.l=n$ acres.

$\therefore\ OE=\frac{2\times 43560\times n}{l}$, and AB, a side of the field $=2OE=\frac{4\times 43560\times n}{l}$.

Hence the area of the field $=\left(\frac{4\times 43560\times n}{l}\right)^2\div 43560=\frac{16\times 43560\times n^2}{l^2}$.

$=43560$ acres. SPINKS.

(side)$^2/160$=number of acres in tract.

$4\times 16\frac{1}{2}\times$side=number of feet in perimeter of field.

$4\left(\frac{4\times 16\frac{1}{2}\times \text{side}}{16}\right)$=number of boards in fence.

$\therefore\ \frac{(\text{side})^2}{160}=\frac{4\,(4\times 16\frac{1}{2}\times\text{side})}{16}=16\frac{1}{2}\times$side.

$\therefore$ side$=2640$ rods$=8\frac{1}{4}$ miles.

$(2640)^2\div 160=43560$ acres. ZERR AND BERG.

(2) *When the field is in the form of a circle.*

Let R=radius in rods.

$\therefore\ \pi R^2/160$=area in acres.

$2\pi R\times 16\frac{1}{2}$=perimeter in feet.

$4\left(\frac{2\pi R\times 16\frac{1}{2}}{16}\right)$=number of boards in perimeter.

$\therefore\ \frac{\pi R^2}{160}=\frac{4(2\pi R\times 16\frac{1}{2})}{16}=8\frac{1}{4}\pi R$.

$\therefore\ R=1320$ rods.

$\pi R^2/160=10890\pi$ acres.

II. Solution by **M. A. GRUBER, A. M., War Department, Washington, D. C.**

Let $16m(p^2+q^2)x$ and $16nx$=the respective sides, in feet, of a *parallelogram*. Then the number of boards in the fence

$$=4\times\tfrac{1}{16}\times 32[m(p^2+q^2)+n]x=8[m(p^2+q^2)+n]x\ldots.(1).$$

For *rectangles*, the number of acres

$$=\frac{(16nx)[16m(p^2+q^2)x]}{43560}=\frac{32mn\,(p^2+q^2)x^2}{5445},$$

which equals (1).

$$\therefore\ x=\frac{5445[m(p^2+q^2)+n]}{4mn(p^2+q^2)}\ldots.(2).$$

For *rhomboids*, the altitude $=16m(p^2-q^2)x$ or $32mpqx$; and number of acres

$$=\frac{32mn\,(p^2-q^2)x^2}{5445}\ \text{or}\ \frac{64mnpqx^2}{5445}.$$

Equating these two values with (1) and reducing, we find

$$x=\frac{5445[m(p^2+q^2)+n]}{4mn(p^2-q^2)}\ldots.(3), \text{ and } x=\frac{5445[m(p^2+q^2)+n]}{8mnpq}\ldots.(4).$$

p, q, and m may be any integers. $p>q$. Also, p and q may be such fractions that, in connection with m and n, will render x integral or of such values as to have the sides divisible by 16.

(a) Take $p=\frac{4}{3}$, $q=\frac{3}{3}$, $m=1$; and substitute in (2), (3), and (4).

Then $x=\frac{5445(1+n)}{4n}\ldots.(5)$, $x=\frac{136125(1+n)}{28n}\ldots.(6)$,

$x=\frac{45375(1+n)}{16n}\ldots.(7)$, and (1) becomes $8(1+n)x$.

x must be integral.

From the factors of 5445, we observe that, in (5), $n=3$, 11, 15, 55, 99, 363, $\frac{1}{3}$, $\frac{5}{3}$, $\frac{3}{5}$, $\frac{11}{5}$, $\frac{5}{11}$, $\frac{9}{11}$, etc.; in (6), $n=55$, 363, $\frac{1}{55}$. etc.; and in (7), $n=15$, $\frac{1}{15}$, etc.

Tne following table of values explains itself.

n	*Eq.*	x	*Boards=Acres*	*Sides in Feet.*
3	(5)	1815	58080	29040 and 87120
11	(5)	1485	142560	23760 and 261360
$\frac{9}{11}$	(5)	3025	44000	48400 and 39600
15	(5)	1452	185856	23232 and 348480
15	(7)	3025	387200	48400 and 726000
55	(5)	1357	620928	22176 and 1219680
55	(6)	4950	2217600	79200 and 4356000
etc.	etc.	etc.	etc.	etc.

(b) Take $p=2$, $q=m=1$; and substitute in (2), (3), and (4).

Then $x=\frac{1089(5+n)}{4n}\ldots.(8)$, $x=\frac{1815(5+n)}{4n}\ldots(9)$, $x=\frac{5445(5+n)}{16n}\ldots(10)$, and (1) becomes $8(5+n)x$.

Whence the following table:

n	*Eq.*	x	*Boards=Acres*	*Sides in Feet*
3	(8)	726	46464	58080 and 34848
3	(9)	1210	77440	96800 and 58080
11	(8)	396	50688	31680 and 69696
11	(9)	660	84480	52800 and 116160
11	(10)	605	77440	48400 and 106480
$\frac{9}{11}$	(8)	1331	61952	106480 and 17424
$\frac{9}{11}$	(10)	2420	112640	193600 and 31680
etc.	etc.	etc.	etc.	etc.

(*c*) And so on for all other values of p, q, and m.

Mr. Gruber sent in three different solutions, and Mr. D. B. Northrup, of Mandana, N. Y., sent in the results for tracts in the form of (1) circle, (2) square, (3) rectangle sides as 2:1, (4) a triangle ratios of side 6:6:7, and (5) an ellipse having a major-axis double the minor-axis. Other very simple solutions of the problem are possible when the tract is in the shape of a square, but the solutions above are quite sufficient for all purposes. ED. F.

136. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

What is the size of the smallest cubical box, inside dimension, that will contain four balls each ten inches in diameter?

Solution by H. N. DAVIS, (Brown University), 159 Brown Street, Providence, R. I.

Let the base of the box be of such a size that two of the balls when placed with their centers along a diagonal will be tangent to each other and to the sides of the box. Let their centers be C and C', the diagonal AD, and the center O.

Draw BC perpendicular to AE. Then $AB=BC=r=5$.

$\therefore AC=5\sqrt{2}$.

$AO=5\sqrt{2}+CO=5\sqrt{2}+r=5\sqrt{2}+5$.

$AD=10(1+\sqrt{2})=\sqrt{2}AE$.

$AE=5\sqrt{2}+10=17.07106$ inches (nearly).

If the second layer of two balls be placed along the other diagonal (at $M+M'$) the position of M with reference to C and the side AF considered as a base will be exactly the same as that of C and C' with reference to AD. The figure may then be taken as an elevation and the height of the box will be exactly equal to its base-edge, and the box will be a cube. Q. E. D.

Good solutions were received from *PROF. J. M. STRASBURG*, Chicago, Ill., *D. B. NORTHRUP*, *G. B. M. ZERR*, and *M. A. GRUBER*.

ALGEBRA.

112. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

In *Hall and Knight's Higher Algebra* I find the following:

If $a+b+c=0$, then

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}; \qquad \frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2};$$

and if $a+b+c+d=0$, then

$$\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^3+b^3+c^3+d^3}{3}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

QUERY. Is there a general law governing such expressions? Investigate.

I. Solution by HARRY S. VANDIVER, Bala, Montgomery County, Pa.

Consider the following general equation with its second term missing, thus

$$f(x)=x^n+m_2x^{n-2}-m_3x^{n-3}\ldots.m_{n-1}x+(-1)^nm_n=0\ldots.(1).$$

Suppose the roots of this equation to be a_1, a_2, a_3, $\ldots.a_n$.

Then $f'(x)=\frac{f(x)}{x-a_1}+\frac{f(x)}{x-a_2}+\frac{f(x)}{x-a^3}\ldots.\frac{f(x)}{x-a_n}\ldots.(2).$

[*Hall and Knight's Higher Algebra*, page 468.]

Divide (1) by $x-a_1$,

$$\frac{f(x)}{x-a_1}=x^{n-1}+a_1x^{n-2}+(m_2+a_1^2)x^{n-3}-(m_3-a_1m_2-a_1^3)x^{n-4}\ldots.;$$

the general term being

$$(-1)^r[m_r-a_1m_{r-1}\ldots.(-1)^ra_1^{r-2}m_2+(-1)^ra_1^r]x^{n-(r+1)};$$

and similar expressions for

$$\frac{f(x)}{x-a_2},\ \frac{f(x)}{x-a_3}\ldots.\frac{f(x)}{x-a_n}.$$

Finding the value of the right-hand member of (2) by addition of the expressions just obtained, and adopting the notation

$$S_h=\frac{a_1^h+a_2^h\ldots.a_n^h}{h},$$

we get, $f'(x)=nx^{n-1}+m_2(n-2)x^{n-3}-m_3(n-3)x^{n-4}\ldots.m_{n-2}x+(-1)^{n-1}m_{n-1}$

$$=nx^{n-1}+(nm_2+2S_2)x^{n-3}-(nm_3-3S_3)x^{n-4}+(nm_4+2m_2S_2+4S_4)x^{n-5}\ldots.;$$

the general term being

$$(-1)^r[nm_r+2m_{r-2}S_2-3m_{r-3}S_3\ldots.(-1)^r(r-2)m_2S_{r-2}+(-1)^rrS_r].$$

Whence, by equating coefficients, and solving for m_2, m_3 $\ldots m_{n-1}$) we obtain,

$$m_2=S_2\ldots.(5);\ m_3=S_3\ldots.(6);\ m_4=\frac{-(2m_2S_2+4S_4)}{4}\ldots.(7);$$

$$m_5=\frac{-(2m_3S_2-3m_2S_3-5S_5)}{5}\ldots.(8);$$

and in general,

$$m_r=\frac{-[2m_{r-2}S_2-3m_{r-3}S_3\ldots.(-1)^r(r-2)m_2S_{r-2}+(-1)^rrS_r]}{r}\ldots.(9).$$

Multiplying (1) by x^{k-n}, we have

$$x^k+m_2x^{k-2}-m_3x^{k-3}\ldots.(-1)^rm_rx^{k-r}\ldots.(-1)^nm_nx^{k-n}=0.$$

Substituting a_1, a_2, $a_3\ldots.a_n$ each separately for x, and adding the identities thus obtained, we have

$$kS_k+(k-2)m_2S_{k-2}-(k-3)m_3S_{k-3}\ldots.(-1)^r(k-r)m_rS_{k-r}\ldots.(-1)^n\ldots.(10).$$

$(k-n)m_nS_{k-n}=0.$

By substituting the value of m_2 in (7) we obtain $S_2^2/2-S_4=m_4$.

In like manner, by successive substitutions, m_5, $m_6\ldots.m_{n-1}$ can be obtained in terms of S_2, $S_3\ldots.S_{n-1}$.

Hence by substitution in (10) we may obtain a relation satisfying the conditions of the problem, for any positive integral value of n.

When $n=3$,

$$kS_k=(k-2)S_2S_{k-2}+(k-3)S_3S_{k-3};$$

putting $k=5$, then $S_5=S_3\times S^2$; when $k=7$, then $S_7=S_5\times S_2$.

. . . . . .

When $n=4$,

$$kS_k=(k-2)S_2S_{k-2}+(k-3)S_3S_{k-3}+(k-4)(S_4-S_2^2/2)S_{k-4};$$

putting $k=5$, $S_5=S_3\times S_2$.

. . . . . .

When $n=5$, we have

$$kS_k-(k-2)S_2S_{k-2}-(k-3)S_3S_{k-3}+(k-4)$$
$$(S_2^2/2-S_4)S_{k-4}-(S_5-S_2S_3)(k-5)S_{k-5}=0.$$

II. Solution by ROBERT B. HAYWARD, Ashcombe, Shanklin, Isle of Wight.

S_r denotes $a^r+b^r+c^r$.

Then $S_1=a+b+c=0.$

$S_2=a^2+b^2+c^2=-2(bc+ca+ab).$

$S_3=a^3+b^3+c^3=3abc.$

Hence a, b, c are the roots of the equation.

$$x^3-\frac{S_2}{2}.x-\frac{S_3}{3}=0.$$

$$\therefore\ x^n-\frac{S_2}{2}.x^{n-2}-\frac{S_3}{3}x^{n-3}=0.$$

Substitute a, b, c successively for x, and add, then

$$S_n - \frac{S_2}{2}.S_{n-2} - \frac{S_3}{3}.S_{n-3} = 0.$$

Whence S_n is determinable in powers of $\frac{S_2}{2}$ and $\frac{S_3}{3}$.

Hence $S_5 = \frac{S_2}{2}.S_3 + \frac{S_3}{3}.S_2 = \frac{5}{6}S_3.S_2$, or $\frac{S_5}{5} = \frac{S_3}{3}.\frac{S_2}{2}$.

$$S_7 = \frac{S_2}{2}.S_5 + \frac{S_3}{3}.S_4 = \frac{S_2}{2}.S_5 + \frac{S_3}{3}\left(\frac{S_2}{7}\right)^2 = 5\left(\frac{S_2}{2}\right)^2.\frac{S_3}{3} + 2.\frac{S_3}{3}\left(\frac{S_2}{2}\right)$$

$$\text{or } \frac{S_7}{7} = \frac{S_5}{5}.\frac{S_2}{2}.$$

It also follows that

$$\frac{S_9}{9} = \frac{S_2}{2}.\frac{S_7}{7} + \tfrac{1}{3}\left(\frac{S_3}{3}\right)^3 \text{ or } \left(\frac{S_2}{2}\right)^3\frac{S_3}{3} + \tfrac{1}{3}\left(\frac{S_3}{3}\right)^3.$$

The form of the expression for $\frac{S_n}{n}$ (putting α_2, α_3 for $\frac{S_2}{2}.\frac{S_3}{3}$) is

$$A\alpha_3^{\,p}\alpha_2^{\,q} + B\alpha_3^{\,p-2}\alpha_2^{\,q+3} + C\alpha_3^{\,p-4}\alpha_2^{\,q+6} + \ldots.$$

continued as far as the indices remain positive.

If n is of the form $6m$, $p=2m$, $q=0$.

$6m-1$, $p=2m-1$, $q=1$.
$6m+1$, $p=2m-1$, $q=2$.
$6m-2$, $p=2m-2$, $q=2$.
$6m+2$, $p=2m$, $q=1$.
$6m-3$, $p=2m-1$, $q=0$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1). Put $a+b+c=0=p$, $ab+ac+bc=q$, $abc=r$.

Then $(1+ax)(1+bx)(1+cx)=1+px+qx^2+rx^3=1+qx^2+rx^3$.

Taking logarithms and equating the coefficients of x^n, we have $\frac{(-1)^{n-1}}{n}(a^n+b^n+c^n)$ for the coefficient equal to the coefficient of x^n in

$$(qx^2+rx^3) - \tfrac{1}{2}(qx^2+rx^3) + \ldots. \pm (1/n)(qx^2+rx^3)^n \ldots.$$

Let $n=2m+1$ and 2, respectively, and we get,

$$\frac{1}{2m+1}(a^{2m+1}+b^{2m+1}+c^{2m+1}) = \pm q^{m-1}r,$$

$$\frac{1}{2m-1}(a^{2m-1}+b^{2m-1}+c^{2m-1})=\mp q^{m-2}r,$$

$$\tfrac{1}{2}(a^2+b^2+c^2)=-q.$$

$$\therefore \frac{a^{2m+1}+b^{2m+1}+c^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+c^{2m-1}}{2m-1}\cdot\frac{a^2+b^2+c^2}{2}.$$

When $m=2, 3$, we get the results in the problem.

(2). Similarly, $(1+ax)(1+bx)(1+cx)(1+dx)=1+qx^2+rx^3+sx^4$.

$\therefore \frac{(-1)^{n-1}}{n}(a^n+b^n+c^n+d^n)$ is equal to the coefficient of x^n in

$$(qx^2+rx^3+sx^4)-\tfrac{1}{2}(qx^2+rx^3+sx^4)^2+\ldots.\pm(1/n)(qx^2+rx^3+sx^4)^n.$$

$\therefore$ As before

$$\frac{a^{2m+1}+b^{2m+1}+c^{2m+1}+d^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+c^{2m-1}+d^{2m-1}}{2m-1}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

The same reasoning will lead to the following :

$$\frac{a^{2m+1}+b^{2m+1}+\ldots.+k^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+\ldots\ +k^{2m-1}}{2m-1}\cdot\frac{a^2+b^2+\ldots.+k^2}{2}.$$

Also solved by *J. M. BOORMAN*, *J. SCHEFFER*, and the *PROPOSER*.

GEOMETRY.

139. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

If $x^2+y^2=1$ [x and y being points corresponding to complex numbers], prove that x and y are at the ends of conjugate radii of an ellipse whose foci are ± 1. [From *Harkness and Morley's Introduction to the Theory of Functions.*]

Solution by J. W. YOUNG, Oliver Fellow in Mathematics, Cornell University, Ithica, N. Y., and FRANK GIFFIN, Assistant in Mathematics, University of Colorado, Boulder, Col.

Let $x=h+ik$. $y=m+in$.

The condition $x^2+y^2=1$. gives on equating real and imaginary parts,

$$h^2+m^2-k^2-n^2=1\ldots.(1),\ hk+mn=0\ldots.(2).$$

Now if the points (h, k) and (m, n) are the extremities of conjugate radii of an ellipse, we may write

$$\left.\begin{matrix} h=a\cos\phi \\ k=b\sin\phi \end{matrix}\right\}\quad\left\{\begin{matrix} m=a\sin\phi \\ n=-b\cos\phi \end{matrix}\right\}\ (a,\ b \text{ semi-axes of ellipse}).$$

These values satisfy condition (2).
Substituting in (1) we have

$$a^2-b^2=1 \ldots . (3).$$

If e is the eccentricity of the ellipse $b^2=a^2(1-e^2)$, substituting in (3) and reducing, we have $ae=\pm 1$, *i. e.* the foci of the ellipse are at the points ± 1.

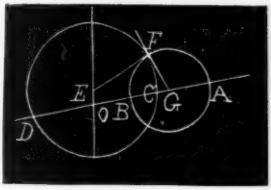
Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

140. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.

Having given two points on a range and a point that bisects the distance between two other points that form an harmonic ratio with the given points, give, if possible, a geometrical construction for locating the other two points.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let A, B be the two given points on the range, O the point bisecting the distance between the other two points. Through O draw OE perpendicular to ABO, and on AB as diameter describe a circle AFB. Draw EF tangent to AFB, and with E as center and a radius equal to EF describe a circle cutting AO in C and D. (F is the point of tangency of EF, and EF must be greater than EO). Then C, D are the two points required.



For $GF^2=AG^2=GC.GD$.
$\therefore$ $AG:GC=GD:AG$.
But $AG+GC:AG-GC=GD+AG:GD-AG$.
$\therefore$ $AC:CB=AD:BD$. Q. E. D.

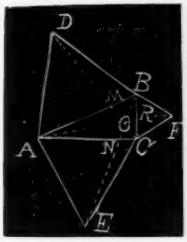
141. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

The equilateral triangle described on the hypotenuse of a right triangle is equivalent to the sum of the equilateral triangles described on the other two sides.

Prove without the aid of the famous Pythagorean proposition.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and NELSON L. RORAY, Bridgeton, N. J.

Let ABC represent the right triangle, and D, E, F the vertices of the equilateral triangles constructed on the three sides.



It is seen at once that $\triangle ACF=\triangle BCE=\frac{1}{2}\triangle ABC$.
$\therefore$ $\triangle ACF+\triangle BCE=\triangle ABC$.
$\triangle BDC=\triangle ABF$, $\triangle ADC=\triangle AEB$.
$\therefore$ $\triangle ABD+\triangle ABC=\triangle ABF+\triangle AEB$.
$\triangle ABC=\triangle ACF+\triangle BCE$.
$\therefore$ $\triangle ABD+2\triangle ABC=(\triangle ABF+\triangle ACF)+(\triangle AEB+\triangle BCE)$.
$\therefore$ $\triangle ABD+2\triangle ABC=\triangle BCF+\triangle ABC+\triangle ACE+\triangle ABC$.

$\therefore \triangle ABD = \triangle BCF + \triangle ACE.$ Q. E. D.

That the three lines AF, BE, and CD intersect in one point may be proved as follows:

Since EOB cuts the sides of $\triangle ACM$ in the points N, O, B, $CN \times OM \times AB = AN \times CO \times BM$....(1).

Since AOF cuts the sides of $\triangle BCM$ in O, P, F, $CO \times BP \times AM = OM \times CP \times AP$....(2).

Multiplying (1) by (2) and canceling, $CN \times BP \times AM = AN \times BM \times CP$.

Also demonstrated by *G. B. M. ZERR*, and the *PROPOSER*.

142. Proposed by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that an infinite number of triangles can be inscribed in $x^2/a^2 + y^2/b^2 - 1 = 0$ whose sides touch $a^2x^2 + b^2y^2 = \frac{a^4b^4}{(a^2+b^2)^2}$.

I. Solution by the PROPOSER.

The curves are $b^2x^2 + a^2y^2 - a^2b^2 = 0$....(1),

and $a^2(a^2+b^2)^2x^2 + b^2(a^2+b^2)^2y^2 - a^4b^4 = 0$....(2).

The invariants of (1) are $\triangle' = -a^4b^4$,

$$\theta' = -a^2b^2(a^4 + a^2b^2 + b^4)^2$$

and of (2), $\triangle = -a^6b^6(a^2+b^2)^4$,

$$\theta' = -2a^4b^4(a^2+b^2)^2(a^4 + a^2b^2 + b^4).$$

By the usual theory, the conditions of the problem are fulfilled if $\theta^2 = 4\triangle\theta'$....(3), which is easily seen to be the case here.

II. Solution by GEORGE A. OSBORNE, Professor of Mathematics, Massachusetts Institute of Technology, Boston, Mass.

The problem is a special case of the following:

An infinite number of triangles can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose sides touch $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$, provided $\frac{a'}{a} + \frac{b'}{b} = 1$.

The problem is considered in Salmon's *Conic Sections*, Art. 376, page 342, for the general form of the conic.

$$S = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

$$S' = a'x^2 + b'y^2 + c'z^2 + 2f'yz + 5g'zx + 2h'xy = 0.$$

The condition that an infinite number of triangles may be inscribed in S and circumscribed about S', is $\Theta'^2 = 4\Delta'\Theta$....(1).

Δ, Δ', are the discriminants of S, S'.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}.$$

$$\Theta = Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh',$$
$$\Theta' = A'a + B'b + C'c + 2F'f + 2G'g + 2H'h,$$

where A, B, C, etc., are minors of Δ, and A', B', C', etc., are minors of Δ'.

For $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ and $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0$, we find $\Delta' = -\frac{1}{a'^2 b'^2}$,

$$\Theta = -\frac{1}{a^2 b'^2} - \frac{1}{a'^2 b^2} - \frac{1}{a^2 b^2}, \quad \Theta' = -\frac{1}{a'^2 b^2} - \frac{1}{a^2 b'^2} - \frac{1}{a'^2 b'^2}.$$

Substituting in (1), we have

$$\left(\frac{a'^2}{a^2} + \frac{b'^2}{b^2} + 1\right)^2 = 4\left(\frac{a'^2}{a^2} + \frac{b'^2}{b^2} + \frac{a'^2 b'^2}{a^2 b^2}\right)$$

from which $\frac{a'}{a} \pm \frac{b'}{b} = \pm 1$.

The only real solution for ellipses is

$$\frac{a'}{a} + \frac{b'}{b} = 1.$$

In problem 142,

$$a' = \frac{ab^2}{a^2 + b^2}, \quad b' = \frac{a^2 b}{a^2 + b^2}, \text{ from which } \frac{a'}{a} + \frac{b'}{b} = 1.$$

A very excellent demonstration was received from *G. B. M. ZERR*.

CALCULUS.

NOTE ON CENTER OF CURVATURE.

By GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The fact, that the point of intersection of two normals which approach each other is neither at infinity nor at the foot of the normals when they become coincident, is not plain to most students. The difficulty may be overcome, in some instances, in the following manner.

Let (x_1, y_1), (x_2, y_2) be two points of the curve; m_1, m_2 the slope of the tangents at these points. The equations of the normals are then

$$x+m_1y=x_1+m_1y_1,\ x+m_2y=x_2+m_2y_2.$$

It is proposed to find the coördinates of the intersection of these lines when (x_1, y_1) approaches and becomes coincident with (x_2, y_2). Eliminating x between the equations, we have

$$y=y_1+\frac{1+m_2\left(\frac{y_1-y_2}{x_1-x_2}\right)}{\frac{m_1-m_2}{x_1-x_2}}.$$

Since $\underset{x_1=x_2}{\text{Lim.}}\left(\frac{y_1-y_2}{x_1-x_2}\right)=\frac{dy_1}{dx_1}$ and $\underset{x_1=x_2}{\text{Lim.}}\left(\frac{m_1-m_2}{x_1-x_2}\right)=\frac{d^2y_1}{dx_1^2}$, we have

$$y=y_1+\frac{1+\left(\frac{dy_1}{dx_1}\right)^2}{\frac{d^2y}{dx^2}},\ \text{and}\ x=x_1-\frac{dy_1}{dx_1}\left\{\frac{1+\left(\frac{dy_1}{dx_1}\right)^2}{\frac{d^2y_1}{dx_1^2}}\right\}.$$

105. Proposed by **CHARLES C. CROSS**, Meridithville, Va.

From all points in a straight line passing through the center of a given circle tangents are drawn to the circle. If the bases and vertices of all the angles thus formed are made to coincide; required the equation of the curve passing through the tangent points.

Solution by **J. W. YOUNG**, Oliver Graduate Scholar in Mathematics, Cornell University, Ithica, N. Y.

Let the circle be $x^2+y^2=a^2$, and the given line $y=0$. Then the length of the tangent from any point $(x_1, 0)$ on the given line is $\sqrt{(x_1^2-a^2)}$.

Also the slope of the tangent may be calculated from the equation $mx_1\pm a\sqrt{(1+m^2)}=0$, which is obtained by substituting the point $(x_1, 0)$ in the tangent equation $y=mx\pm a\sqrt{(1+m^2)}$. Solving for m, we have

$$m^2=\frac{a^2}{x_1^2-a^2},\ \text{or}\ m=\pm\frac{a}{\sqrt{(x_1^2-a^2)}}.$$

To determine the required locus, use polar coördinates, with the common vertex of angles as pole and their common base as initial line; the coördinates (r, θ) of the tangent points are then given by the equations

$$\left.\begin{aligned} r&=\sqrt{(x_1^2-a^2)}\\ \tan\theta&=\frac{\pm a}{\sqrt{(x_1^2-a^2)}}\end{aligned}\right\}$$

Hence, the required locus is $r\tan\theta=\pm a$; in Cartesian coördinates this becomes $x^2(y^2-a^2)+y^4=0$.

Also solved by *H. C. WHITAKER*, and *G. B. M. ZERR*.

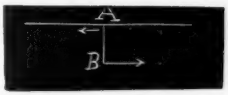
MECHANICS.

102. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A heavy particle with a light string attached is placed on the edge of a smooth table. A boy, holding the string horizontally, runs at right angles to the string. Determine the motion of the particle (1) when the boy runs with a uniform velocity; (2) when he runs with a uniform acceleration.

Solution by the PROPOSER.

This problem is one of a class in which a clear idea of the motion may be gained without solving analytically.



Let A be the position of the particle at the start, B the position of the boy's hand. We shall suppose the boy runs to the right. Now the effect upon the motion of the particle will be the same if, instead of supposing the boy to run to the right, we impose upon space a hypothetical motion to the left, in such a way as to bring the boy's hand at B at rest. This of course necessitates imposing upon the particle at A the same motion that is imposed upon space.

In our first case thus we impose upon A a uniform velocity to the left. We have thus a particle acted upon by gravity, and moving about a fixed point with uniform velocity, *i. e.* a simple conical pendulum. So the result to (1) is that as the boy runs with uniform velocity the particle will move around his hand as a simple conical pendulum.

In our second case impose upon A a constant acceleration to the left. Our particle is thus acted on by two forces, constant in direction and magnitude, viz. gravity and this hypothetical horizontal force. It will therefore oscillate horizontally and vertically about B. Also since these two forces may be combined into a single oblique force constant in magnitude and direction, these two oscillations will be equivalent to a single simple oscillation obliquely about B. The result to (2) then, is that as the boy runs with uniform acceleration the particle instead of moving around his hand as a conical pendulum, will oscillate obliquely behind it.

103. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Given the lengths a, b of the sides of a parallelogram, the direction of side a, and the position of the centroid. Prove that the locus of the foci of the ellipse of gyration at the centroid is a Cassinian Oval, having its foci distant $a/2\sqrt{3}$ from the centroid, and the constant product of its focal distances equal to $\frac{1}{12}b^2$.

Solution by the PROPOSER.

Let A, B be the principal moments of inertia, m the mass of the parallelogram, then $x^2/A+y^2/B=1/m$ is the equation to the ellipse of gyration at the centroid.

$$\text{Eccentricity} = \sqrt{\frac{A-B}{A}}, \quad \text{semi-major axis} = \sqrt{\frac{A}{m}}.$$

$\therefore$ Distance of focus from center$=\sqrt{\frac{A-B}{m}}$.

Let θ=the angle the principal axes make with the sides. then if u, v be the coördinates of the focus, we easily get

$$u=\sqrt{\frac{A-B}{m}}\cos\theta,\quad v=\sqrt{\frac{A-B}{m}}\sin\theta.$$

$\therefore u^2+v^2=\frac{A-B}{m}$, $u^2-v^2=\frac{A-B}{m}\cos 2\theta$.

From problem 94, solution on page 48, Vol. VII, No. 2, we get

$$u^2+v^2=\tfrac{1}{12}V[a^4+b^4+2a^2b^2\cos 2\beta].$$
$$u^2-v^2=\tfrac{1}{12}(a^2+b^2\cos 2\beta).$$

Eliminating $\cos 2\beta$ we get

$$144(u^2+v^2)^2=24a^2(u^2-v^2)-a^4+b^4.$$

Let $u=r\cos\varphi$, $v=r\sin\varphi$.

$\therefore 144r^4=24a^2r^2\cos 2\varphi-a^4+b^4$, or $r^4=\frac{1}{6}a^2r^2\cos 2\varphi-a^4/144+b^4/144$.

If $a=b$, $r^2=\frac{1}{6}a^2\cos 2\varphi$.

104. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

From a locomotive and tender standing still on a bridge, the pressure on the bridge is p_1=80 tons. The track is supposed to be straight and practically horizontal. Had the locomotive and tender been running at the rate of r=60 miles an hour, how many tons would the pressure on the bridge have been?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

p_1=80 tons$=W=mg$.

Both m and g are constant.

$\therefore$ The pressure is the same, 80 tons, no matter what the velocity.

DIOPHANTINE ANALYSIS.

83. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find three numbers in arithmetical progression whose sum is a square and cube.

I. Solution by J. W. YOUNG, Cornell University, Ithica, N. Y.; B. L. REMICK, Bradley Polytechnic Institute, Peoria, Ill.; and ALOIS F. KOVARIK, Decorah Institute, Decorah, Ia.

A number which is a square and a cube is a sixth-power.

Also three numbers in arithmetical progression may be represented by

$$a-d.\ a,\ a+d;\ \text{whose sum is } 3a.$$

These considerations lead at once to the following expressions for the required numbers : 3^5x^6-d, 3^5x^6, 3^5x^6+d, where x, d, are any numbers.

The sum is evidently $(3x)^6=[(3x)^2]^3=[(3x)^3]^2$.

As an example we may take $x=1$, $d=100$.

$$143,\ 243,\ 343,\ \text{whose sum}=729=27^2=9^3.$$

II. Solution by the PROPOSER.

Let $\frac{1}{3}(x-y)^2$, $\frac{1}{3}(x^2+y^2)$, $\frac{1}{3}(x+y)^2$ be the three numbers.

Their sum is x^2+y^2.

Let $x^2+y^2=a^6m^6$. Let $x=m^2-n^2$, $y=2mn$.

$\therefore\ x^2+y^2=(m^2+n^2)^2=a^6m^6$.

$\therefore\ m^2+n^2=a^3m^3$.

Let $n=pm$.

$\therefore\ m^2(1+p^2)=a^3m^3$.

$$\therefore\ m=\frac{1+p^2}{a^3},\quad n=\frac{p(1+p^2)}{a^3}.$$

$$x=\frac{(1+p^2)^2(1-p^2)}{a^6},\quad y=\frac{2p(1+p^2)^2}{a^6},\quad x^2+y^2=\frac{(1+p^2)^6}{a^{12}}.$$

$\therefore$ The numbers are

$$\frac{1}{3}\left(\frac{(1+p^2)^2(1-2p-p^2)}{a^6}\right)^2,\quad \frac{1}{3}\left(\frac{(1+p^2)^6}{a^{12}}\right),\quad \frac{1}{3}\left(\frac{(1+p^2)^2(1+2p-p^2)}{a^6}\right)^2.$$

Also solved by *CHAS. C. CROSS, JOSIAH H. DRUMMOND, M. A. GRUBER, J. SCHEFFER*, and *ELMER SCHUYLER*.

84. Proposed by the late SYLVESTER ROBINS, North Branch Depot, N. J.

The nth term of an infinite series of "nests" contains all the prime, integral, rational parallelopipeds that have 3^n for their solid diagonals. It is required to determine the general expression for N=the number of such solids in nth term, and to exhibit the dimensions of all the "eggs" in the first six nests.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In parallelopipeds the square of the solid diagonal equals the sum of the squares of the three dimensions (length, breadth, and height). Whence, $(3^n)^2=$ the sum of three squares.

When $(3^n)^2=$the sum of three *integral* squares, I have found, by inspection, that the entire number of sets of three squares is, in terms of n, $\frac{3^n+2n-1}{4}$, of which $\frac{3^{n-1}+1}{2}$ are prime sets, and $\frac{3^{n-1}+2n-3}{4}$ multiple sets.

$\therefore$ According to the problem, $n=\frac{3^{n-1}+1}{2}$.

In *prime* sets of the sum of three squares equal to a square, it will be observed that two of the three squares are even and the other odd.

By means of an extensive table containing the odd numbers that are equal

to the sum of two squares, I have found the following complete sets of "eggs" for each nest.

In nest 1, where $n=1$ and $N=1$: 1, 2, 2. PROOF.—$1^2+2^2+2^2=(3^1)^2=9$.

In nest 2, where $n=2$ and $N=2$: 1, 4, 8; 7, 4, 4.

In nest 3, where $n=3$ and $N=5$: 7, 2, 26; 7, 14, 22; 23, 2, 14; 23, 10, 10; 25, 2, 10.

In nest 4, where $n=4$ and $N=14$:

1, 28, 76; 1, 44, 68; 17, 56, 56; 23, 16, 76; 23, 44, 64; 41, 16, 68; 41, 28, 64; 47, 16, 64; 49, 8, 64; 49, 32, 56; 55, 20, 56; 55, 40, 44; 65, 20, 44; 79, 8, 16.

In nest 5, where $n=5$ and $N=41$:

1, 22, 242; 17, 14, 242; 17, 46, 238; 17, 106, 218; 17, 134, 202; 31, 38, 238; 31, 158, 182; 47, 14, 238; 47, 154, 182; 49, 2, 238; 49, 158, 178; 95, 118, 190; 95, 50, 218; 95, 130, 182; 97, 46, 218; 97, 62, 214; 97, 94, 202; 97, 134, 178; 113, 22, 214; 113, 62, 206; 113, 74, 202; 113, 146, 158; 127, 22, 206; 127, 46, 202; 127, 106, 178; 127, 134, 158; 143, 50, 190; 143, 74, 182; 143, 122, 154; 145, 70, 182; 161, 2, 182; 161, 38, 178; 193, 22, 146; 193, 62, 134; 193, 70, 130; 209, 22, 122; 209, 38, 118; 223, 22, 94; 223, 62, 74; 239, 22, 38; 241, 22, 22.

In nest 6, where $n=6$ and $N=122$:

17, 136, 716; 17, 436, 584; 49, 236, 668; 49, 224, 692; 49, 128, 716; 49, 496, 532; 55, 196, 700; 73, 116, 716; 73, 44, 724; 79, 32, 724; 79, 172, 704; 79, 112, 716; 79, 308, 656; 79, 340, 640; 79, 460, 560; 89, 224, 688; 89, 416, 592; 103, 376, 616; 103, 424, 584; 119, 68, 716; 119, 196, 692; 119, 436, 572; 119, 484, 532; 137, 4, 716; 137, 236, 676; 143, 124, 704; 143, 284, 656; 175, 196, 680; 175, 104, 700; 185, 40, 704; 185, 296, 640; 199, 136, 688; 199, 248, 656; 199, 304, 632; 199, 376, 592; 239, 308, 616; 241, 4, 688; 241, 416, 548; 241, 128, 676; 241, 464, 508; 247, 116, 676; 247, 236, 644; 271, 32, 676; 271, 220, 640; 271, 208, 644; 271, 380, 560; 313, 56, 656; 313, 304, 584; 329, 92, 644; 329, 460, 460; 337, 56, 644; 337, 196, 616; 359, 56, 632; 359, 152, 616; 359, 248, 584; 359, 424, 472; 367, 236, 584; 367, 296, 556; 401, 172, 584; 401, 248, 556; 401, 296, 532; 401, 364, 488; 409, 376, 472; 409, 152, 584; 431, 68, 584; 431, 136, 572; 431, 296, 508; 431, 376, 452; 439, 172, 556; 439, 196, 548; 439, 236, 532; 439, 284, 508; 455, 104, 560; 455, 280, 496; 457, 116, 556; 457, 364, 436; 473, 304, 464; 497, 196, 496; 497, 224, 484; 521, 44, 508; 521, 100, 500; 521, 220, 460; 521, 236, 452; 521, 340, 380; 527, 196, 464; 527, 284, 516; 529, 40, 500; 529, 116, 488; 529, 200, 460; 529, 268, 424; 529, 288, 436; 529, 332, 376; 551, 112, 464; 551, 304, 368; 583, 136, 416; 583, 224, 376; 593, 4, 424; 593, 196, 376; 623, 44, 376; 623, 104, 364; 623, 236, 296; 625, 196, 320; 625, 220, 304; 631, 28, 364; 631, 196, 308; 649, 4, 332; 649, 124, 308; 649, 172, 284; 649, 196, 268; 655, 4, 320; 655, 100, 304; 655, 40, 296; 655, 104, 280; 689, 32, 236; 689, 116, 208; 695, 4, 220; 695, 100, 196; 719, 32, 116; 719, 44, 112; 721, 28, 104; 721, 40, 100; 721, 56, 92.

In order to obtain the multiple sets in the several nests, we multiply the prime sets of all the preceding nests by the necessary power of 3.

The multiple sets in nest 4 are 3^3 times the prime set of nest 1, 3^2 times the prime sets of nest 2, and 3 times those of nest 3, making 8 in all.

Also solved by *CHARLES C. CROSS.*

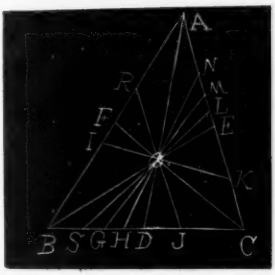
AVERAGE AND PROBABILITY.

56. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find the chance that the center of gravity of a triangle lies inside the triangle formed by three points taken at random within the triangle. [From *Williamson's Integral Calculus.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let ABC be the given triangle, D, E, F the mid-points of the sides, and O the center of gravity. Also let $DG=x$, DS, DH, DJ, FK, FL, FN, ER, or $EI=y$.



Now $FM=BM-\frac{1}{2}c=\dfrac{c(a-2x)}{2(a+6x)}$.

When the first point is on OG and the second on OH, OS, OJ, OK, OL, ON, OR, or OI, respectively, then the third point must fall on

$$MON=\frac{\triangle}{3}\left(\frac{4x}{a+6x}-\frac{4y}{a+6y}\right),\quad MOL=\frac{\triangle}{3}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right),$$

$$MARO=\frac{\triangle}{3}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right),\quad MAIO=\frac{\triangle}{3}\left(\frac{4x}{a+6x}+1-\frac{4y}{c+6y}\right),$$

$$MACSO=\frac{\triangle}{3}\left(\frac{4x}{a+6x}+1+\frac{4y}{c+6y}\right),\quad MBHO=\frac{\triangle}{3}\left(2-\frac{4x}{a+6x}-\frac{4y}{c+6y}\right),$$

$$MOJB=\frac{\triangle}{3}\left(\frac{4y}{b+6y}+1-\frac{4x}{a+6x}\right),\quad MOK=\frac{\triangle}{3}\left(1-\frac{4y}{b+6y}-\frac{4x}{a+6x}\right),$$

respectively.

The chance that the first is on OG is $dx/3a$; that the second is on OH, OS, or OJ is $dy/3a$; that it is on OK, OL, or ON is $dy/3c$; that it is on OR, or OI is $dy/3b$.

$$\therefore p=\frac{1}{27a^2}\left[\int_0^{\frac{1}{2}a}\left\{\int_0^{x}\left(\frac{4x}{a+6x}-\frac{4y}{a+6y}\right)dy+\int_x^{\frac{1}{2}a}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right)dy\right\}dx\right.$$

$$\left.+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}a}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right)dxdy\right]+\frac{1}{27ac}\left[\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}\left(\frac{4x}{a+6x}+1-\frac{4y}{c+6y}\right)dxdy\right.$$

$$\left.+\int_0^{\frac{1}{2}a}\left\{\int_0^{FM}\left(\frac{4x}{a+6x}+1+\frac{4y}{c+6y}\right)dy+\int_{FM}^{\frac{1}{2}c}\left(2-\frac{4x}{a+6x}-\frac{4y}{c+6y}\right)dy\right\}dx\right]$$

$$+\frac{1}{27ab}\left[\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}b}\left(\frac{4y}{b+6y}+1-\frac{4x}{a+6x}\right)dxdy+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}b}\left(1-\frac{4y}{b+6y}-\frac{4x}{a+6x}\right)dxdy\right]$$

$$\therefore p=\frac{1}{27a}\int_0^{\frac{1}{2}a}\left[\frac{5}{2}+\frac{a-34x}{6(a+6x)}+\frac{4x(a-2x)}{(a+6x)^2}+\frac{4}{9}\log\left(\frac{a+6x}{a}\right)-\frac{8}{9}\log 4\right]dx$$

$$=\tfrac{1}{27}\left(\tfrac{1}{8}+\tfrac{8}{9}\log 4\right).$$

. Since the first point can fall in any of the six portions into which the medians divide the triangle, the required probability is $P=6p$.

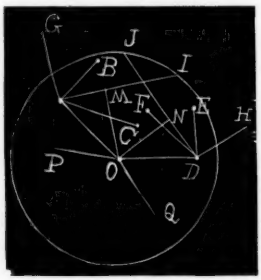
$$\therefore P=\tfrac{6}{2^7}(\tfrac{1}{3}+\tfrac{5}{9}\log 4)=\tfrac{1}{2^7}(2+\tfrac{10}{3}\log 4=\tfrac{2}{2^7}(1+\tfrac{10}{3}\log 2).$$

91. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Six points A, B, C, D, E, F, are taken at random on the surface of a sphere. Find the chance that the plane through A, B, C intersects the plane through D, E, F within the sphere.

Solution by the PROPOSER.

Let AI, DJ be the diameters of the sections of the sphere made by the planes through A, B, C and D, E, F; M and N their centers, O the center of the sphere, OP a line such that AB is parallel to the plane MOP, OQ a line such that DE is parallel to the plane NOQ. Draw AG, DH perpendicular to AI, DJ, respectively.



Let $AO=r$, $\angle AOM=\theta$, $\angle DON=\varphi$, $\angle GAC=\psi$, $\angle GAB=\delta$, $\angle HDF=\mu$, $\angle HDE=\nu$, $\angle MOP=\lambda$, the angle the plane MOP makes with some fixed plane through $OP=\rho$, $\angle NOQ=\gamma$, and the diedral angle $MOQN=\eta$.

An element of surface at A is $4\pi r^2\sin\theta d\theta$; at D, $4\pi r^2\sin\varphi d\varphi$; at C, $4r^2\sin\theta\sin\psi d\psi d\lambda$; at B, $4r^2\sin\theta\sin(\psi-\delta)\sin\lambda\sin\delta d\delta d\rho$; at F, $4r^2\sin\varphi\sin\mu d\mu d\gamma$; at E, $4r^2\sin\varphi\sin(\mu-\nu)\sin\gamma\sin\nu d\nu d\eta$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and $\frac{1}{2}\pi$; of ψ, 0 and π; of δ, 0 and ψ; of μ, 0 and π; of ν, 0 and μ; of λ, 0 and π; of γ, $\pm(\theta-\varphi)$ and $\theta+\varphi$ (the double sign being taken + when $\theta>\varphi$, and − when $\theta<\varphi$); of ρ, 0 and 2π; of η, 0 and 2π.

Since the whole number of ways six points can be taken in the surface of the sphere is $(4\pi r^2)^6$ the required chance is

$$p=\frac{1}{(4\pi r^2)^6}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\int_{\pm(\theta-\psi)}^{\theta+\psi}\int_0^{2\pi}\int_0^{2\pi}4\pi r^2\sin\theta.d\theta 4\pi r^2$$

$$\sin\varphi d\varphi.4r^2\sin\theta\sin\psi d\psi d\lambda.4r^2\sin\varphi\sin\mu d\mu d\gamma.4r^2\sin\theta\sin(\psi-\delta)\sin\lambda\sin\delta d\delta d\rho$$

$$\times 4r^2\sin\varphi\sin(\mu-\nu)\sin\gamma\sin\nu d\nu d\gamma$$

$$=\frac{2}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\int_{\pm(\theta-\psi)}^{\theta+\psi}\int_0^{2\pi}\sin^3\theta\sin^3\varphi\sin\psi\sin\mu\sin(\psi-\delta)$$

$$\sin(\mu-\nu)\sin\delta\sin\nu\sin\lambda\sin\gamma d\theta d\varphi d\psi d\mu d\delta d\nu d\gamma d\rho$$

$$=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\int_{\pm(\theta-\psi)}^{\theta+\psi}\sin^3\theta\sin^3\varphi\sin\psi\sin\mu\sin(\psi-\delta)$$

$$\sin(\mu-\nu)\sin\delta\sin\nu\sin\lambda\sin\gamma d\theta d\varphi d\psi d\mu d\delta d\nu d\lambda d\gamma$$

$$=\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\int_0^{\pi}\sin^4\theta\sin^4\varphi\sin\psi\sin\mu\sin(\psi-\delta)\sin(\mu-\nu)$$

$$\sin\delta\sin\nu\sin\lambda d\theta d\varphi d\psi d\mu d\delta d\nu d\lambda$$

$$=\frac{16}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\int_0^{\mu}\sin^4\theta\sin^4\varphi\sin\psi\sin\mu\sin(\psi-\delta)\sin(\mu-\nu)\sin\delta\sin\nu$$

$$d\theta d\varphi d\psi d\mu d\delta d\nu$$

$$=\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\int_0^{\psi}\sin^4\theta\sin^4\varphi\sin\psi\sin\mu(\sin\mu-\mu\cos\mu)\sin(\psi-\delta)\sin\delta$$

$$d\delta d\varphi d\psi d\mu d\delta$$

$$=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\int_0^{\pi}\sin^4\theta\sin^4\varphi\sin\psi(\sin\psi-\psi\cos\psi)\sin\mu(\sin\mu-\mu\cos\mu)$$

$$d\theta d\varphi d\psi d\mu$$

$$=\frac{3}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\pi}\sin^4\theta\sin^4\varphi\sin\psi(\sin\psi-\psi\cos\psi)d\theta d\varphi d\psi$$

$$=\tfrac{3}{4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^4\theta\sin^4\varphi d\theta d\varphi=\frac{27\pi}{64}\int_0^{\frac{1}{2}\pi}\sin^4\theta d\theta=\frac{81\pi^2}{1024}=(\tfrac{3}{4})^4(\tfrac{1}{2}\pi)^2.$$

NOTE. We may publish a second solution of Problem 90 in the next issue of the MONTHLY. ED. F.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

140. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

$\frac{1}{7}=0.\dot{1}4285\dot{7}$; $\frac{1}{14}=0.0\dot{7}14285$; $\frac{1}{21}=0.0\dot{0}4761\dot{9}$. Notice that the sum of the figures in each period is equal to 27. This is not true with $\frac{1}{72}$, $\frac{1}{73}$. Is there any general law of which these are special cases, and if so, what is it?

141. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

If the alloy in a half-dollar be 1-13th of the mass, and the coin be worth a cent if it be all alloy, what should be the exact value of the coin if it be all pure silver?

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

ALGEBRA.

129. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Prove that

$$\begin{vmatrix} 1-\binom{m-1}{0} & -\binom{m-1}{1} & \cdots & \binom{m-1}{m-2} \\ 1 & 1 & -\binom{m-2}{0} & \cdots \binom{m-2}{m-3} \\ 1 & 0 & 1 & \cdots \binom{m-3}{m-4} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 1 \quad \binom{1}{0} \\ 1 & 0 & 0 \ldots 0 & 1 \end{vmatrix} \quad e=\sum_{n=\infty}^{n=1}\frac{n^m}{n!}, \text{ where } \binom{m-2}{k}=\frac{(n-2\ldots(m-k-1)}{k!}.$$

130. Proposed by J. M. BOORMAN, Woodmere, N. Y.

Solve $x^5-y^5=2101\ldots.(1)$, $x-y=1\ldots.(2)$.

Find general formula for (1), ... (2), when $x^n-y^n=a$; for $n_0=3$; $n_1=5$; $n_2=7$; etc.

131. Proposed by HARRY S. VANDIVER, Bala, Montgomery County, Pa.

It is well known that, when we define the symbol $\sqrt[n]{a}$ after the manner of elementary text-books on algebra, certain *irrational equations* may be written down which have no real or imaginary roots. Required then, the condition, if any, between a, b, c, and d such that the equation, $ax+b+\sqrt{(cx^2+d)}=0$, shall have no root, real or or imaginary.

*** Solutions of these problems should be sent to J. M. Colaw not later than April 10.

GEOMETRY.

146. Proposed by H. R. HIGLEY, M. Sc., Professor of Mathematics, Normal School, East Stroudsburg, Pa.

If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square on the line joining the points of concurrence=the sum of the squares on the two tangents from these points. Ex. 24, page 219, Mackay's *Elements of Euclid*.

147. Proposed by R. A. WELLS, Professor of Mathematics, Franklin College, New Athens, Ohio.

Find the locus in space of the point which is equally illuminated by each of two unequal lights whose intensities are a and b ($a>b$), placed at a distance c from each other.

160. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let GFH be the spherical triangle formed by joining the mid-points of the sides of the spherical triangle ABC; E the spherical excess of ABC; β, p the base and altitude of GFH. Prove $\sin\frac{1}{2}E=\sin\beta\sin p$.

*** Solutions of these problems should be sent to B. F. Finkel not later than April 10.

CALCULUS.

123. Prize Problem. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find in finite terms, the value of $\int_0^{\frac{1}{2}\pi}\log\tan\phi\,d\phi$.

A year's subscription to the MONTHLY will be given to the person sending to the Proposer the first solution of this problem. This problem was misstated in last issue.

124. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Show that the cardioides $r=a(1+\cos\theta)$....(1), and $r=b(1-\cos\theta)$....(2), intersect at right angles.

125. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Show that the *complete primitive* of the Differential Equation

$$\left[\tan^{-1}(x)-\frac{x}{1+x^2}\right]\frac{d^2y}{dx^2}=2\left(\frac{x}{(1+x^2)^2}\right)\left[x\frac{dy}{dx}-y\right],$$

is $y=C\tan^{-1}(x)+cx$.

*** Solutions of these problems should be sent to J. M. Colaw not later than April 10.

EDITORIALS.

Dr. C. N. Little, professor of Mathematics in Leland Stanford University, has resigned his position.

Mr. S. W. Reaves, graduate scholar in Cornell University, has been appointed instructor in Mathematics at Orchard Lake Military Academy.

Professor Charles Hermite, the venerable dean of French mathematicians, died after a brief illness at his home in Paris, on January 14, 1901, and in his death the mathematical world sustains a great loss. He was born at Dieuze, December 25, 1822. In 1858, 1865, 1866, his transcendental solution of the quintic equation involving elliptic integrals was published in the *Comptes Rendus*. The theory of Differential Equations, the reduction of Abelian to Elliptic Functions, the Theory of Functions, and many other mathematical subjects, have received substantial additions at the hands of this great savant.

BOOKS AND PERIODICALS.

The Teaching of Mathematics in the Higher Schools of Prussia. By J. W. A. Young, Ph. D., Assistant Professor of the Pedagogy of Mathematics in the University of Chicago. 8vo. Cloth, 141 pages. Price, 80 cents. New York: Longmans, Green & Co.

The account of the Prussian High School System with detailed and specific description of the work in mathematics as set forth in this little volume is most timely—coming as it does at the close of the 19th century, the last part of the last twenty-five years of which has witnessed great changes and improved methods in mathematical teaching in America. There is still room for great improvement along the line and just at present great pressure is being brought to bear on mathematical teaching in colleges by the great universities, the colleges in turn are demanding a better quality of work in mathematics of the high school, and it is hoped that the result will be general improvement all along the line of mathematical teaching.

Dr. Young has gathered much of the material for the account respecting the Prussian Higher School System by personal observation, and in this account are to be found much that is of highest value to American teachers. All teachers in America who have been the subjects of political intrigue and personal whims rejoice to know that the German teacher works with a sense of security in his position without regard to political occurrences, or the whims of the powerful and influential, security in a modest compe-

tency while at work, security in his profession as a life work. This enables the German teacher to utilize all his energies in the improvement of his mind, and thus none are lost in the way of political scheming so common and so debasing in this country.

The American Journal of Mathematics. Edited by Frank Morley with the coöperation of Simon Newcomb, S. Cohen, Charlotte A. Scott and other mathematicians. Vol. XXIII, No. 1.

This number contains the following papers:

Die Typen der linearen Complexe rationaler Curven im Rr., Von S. Kantor; Transformations of Systems of Linear Differential Equations, by E. J. Wilczynski; Distribution of the Ternary Linear Homogenious Substitutions in Galios Field into Complete Sets of Conjugate Substitutions, by L. E. Dickson; Distribution of the Quarternary Linear Homogenious Substitution in a Galois Field into Complete Sets of Conjugate Substitutions, by T. M. Putnam; On the Determination and Solution of the Metacyclic Quintic Equations with Rational Coefficients, by J. C. Glashan; Construction of the Geometry of Euclidean n-Dimensional Space by the Theory of Continuous Groups, by E. O. Lovett; A Table of Class Numbers for Cubic Number Fields, by L. W. Reid; On Certain Properties of the Plane Cubic Curve in Relation to the Circular Points at Infinity, by R. A. Roberts.

The Annals of Mathematics. Published in October, January, April, and July, under the auspices of Harvard University. Price, $2.00 per year in advance.

Among the articles in the January number for 1901 are the following: An Application of Elliptic Functions to Peaucellier's Link-Work, by Dr. Arnold Emch; Note on the Geometrical Treatment of Conics, by Professor Charlotte A. Scott; On a Special Class of Abelian Groups, by Dr. G. A. Miller; The Theory of Linear Dependence, by Maxime Bocher.

Divergent and Conditionally Convergent Series Whose Product is Absolutely Convergent. By Dr. Florian Cajori.

This is a reprint of a very interesting article which appeared in the Transactions of the American Mathematical Society.

ERRATA.

Pages 2—7, all square brackets should be parenthesis.

Pages 3—4, for H read K [kappa].

Page 4, first line, $(x-r)\lambda-1$ should read $((x-r)\lambda-1)$.

Page 5, in formula (14), $\binom{K+1}{\nu^1}$ should read $\binom{\nu^1}{K+1}$.

Page 7, line 6 from below $\begin{bmatrix}\nu^2\\ r\end{bmatrix}$ should be $\binom{\nu}{r}$.

Page 21, see the restatement of Problem 129, Algebra, in this number.

Page 22, see the restatement of Prize Problem, 123, Calculus, on page 54, of this number.

Page 24, line 5, for π read e.

Page 235, Vol. VII, Problem 108, Mechanics, should be numbered 109. Renumber accordingly, all the problems in Mechanics, proposed since then.

Problems 146 and 147, Geometry, in this issue, are proposed for the May number, Vol. VII.

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ON THE PRODUCT OF TWO COMMUTATIVE OPERATORS.

By DR. G. A. MILLER, Cornell University, Ithaca, New York.

The aim of this note is to give a very elementary explanation of the order of the product of two commutative operators. Such an explanation seems the more desirable on account of the fact that an error in regard to this matter occurs in one of the best known works on the theory of groups.* We shall first consider the cases when the orders of the two operators (s_1, s_2) are powers of the same prime number (p), the order of s_1 being p^α and that of s_2 being p^β $(\alpha \gtreqless \beta)$.

From the equation $(s_1 s_2)^n = s_1{}^n s_2{}^n$ and the fact that $s_1{}^n s_2{}^n = 1$ only when $s_1{}^n = s_2{}^{-n}$, it follows that $s_1 s_2$ is of the same order as s_1 whenever $\alpha > \beta$. When $\alpha = \beta$ several cases present themselves: (1) The groups generated by s_1 and s_2 have only identity in common. In this case it follows from the given equations that $s_1 s_2$ is of the same order as s_1, viz., of order p^α. (2) s_1 is a power of s_2; *e. g.* $s_1 = s_2{}^\gamma$. In this case the order of $s_1 s_2$ may be any power of p from p^0 to p^α when $p > 2$, and p^0 to $p^{\alpha-1}$ when $p = 2$. These orders may be obtained by assigning the following values to γ:

$$p^\alpha - 1,\ p^{\alpha-1} - 1,\ p^{\alpha-2} - 1,\ \ldots . p - 1,\ 1.$$

In general, let $s_1 p^{\alpha_1}$ be the first power of s_1 which is also a power of s_2, so that $s_1{}^{p^{\alpha_1}} = s_2{}^{kp^{\alpha_1}}$ (α_1 does not equal α, and k being prime to p and less than $p^{\alpha-\alpha_1}$). It follows from the first sentence of the preceding paragraph that the

*Burnside, *Theory of Groups of a Finite Order*, 1897, page 16.

order of $s_1 s_2$ cannot be less than p^{α_1} nor greater than p^{α}. If we assign to k the following values:

$$p^{\alpha-\alpha_1}-1,\ p^{\alpha-\alpha_1-1}-1,\ p^{\alpha-\alpha_1-2}-1,\ \ldots . p-1,\ 1\ (\alpha_1 \text{ does not equal } \alpha),$$

we observe that $s_1 s_2$ may have for its order any power of p from p^{α_1} to p^{α} when p is odd (or when $p=2$ and $\alpha=\alpha_1$), and from p^{α_1} to $p^{\alpha-1}$ when $p=2$ and $\alpha>\alpha_1$.

These results are expressed by the following

THEOREM. *It is possible to find two commutative operators* (s_1, s_2) *of the same order* (p^{α}) *such that* $s_1 p^{\alpha_1} = s_2 k p^{\alpha_1}$ $(\alpha \gtrless \alpha)$ *and* $s_1 s_2$ *is of order* p^{δ}, *where* δ *can have any value from* α_1 *to* α *when* $p>2$ *(or when* $p=2$ *and* $\alpha=\alpha_1$*), and from* α_1 *to* $\alpha-1$ *when* $p=2$ *and* $\alpha>\alpha_1$.

When the orders of s_1 and s_2 are not powers of the same prime they may be represented by $2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \ldots .$ and $2^{\beta_0} p_1^{\beta_1} p_2^{\beta_2} \ldots .$ respectively; $p_1, p_2, \ldots .$ being odd prime numbers, and the exponents being positive integers including 0. We may think of s_1 and s_2 as the products of operators of orders 2^{α_0}, $p_1^{\alpha_1}$, $p_2^{\alpha_2}$, ... and of orders 2^{β_0}, $p_1^{\beta_1}$, $p_2^{\beta_2}$, ... respectively. $s_1 s_2$ is then the product of all of these operators. Combining the pairs which are powers of the same prime, $s_1 s_2$ may be represented as the product of operators of orders 2^{γ_0}, $p_1^{\gamma_1}$, $p_2^{\gamma_2}$, ... ($p_x^{\gamma_x}$ being the product of the given operators of orders $p_x^{\alpha_x}$ and $p_x^{\beta_x}$). Since the groups generated by these operators have only identity in common $s_1 s_2$ is of order $2^{\gamma_0} p_1^{\gamma_1} p_2^{\gamma_2} \ldots .$ We have now reduced this case to one of the earlier ones.

It was observed in the second paragraph that γ_x is equal to the larger of the two numbers α_x and β_x whenever these are different, and that γ_x may be 0 whenever $\alpha_x=\beta_x$. *Hence the minimum order* (m) *of* $s_1 s_2$ *is the product of the highest of the primes which divide one and only one of their orders.** The maximum order (M) of $s_1 s_2$ is clearly the lowest common multiple of their orders. From the given theorem it follows that s_1 and s_2 can be so selected that the order of $s_1 s_2$ is any factor of M which is not less than m.

HYDRAULIC SOLUTION OF AN ALGEBRAIC EQUATION OF THE nth DEGREE.

By DR. ARNOLD EMCH, University of Colorado.

In the January (1901) number of this MONTHLY I have established two methods of extracting the nth root of any positive real number. In conclusion, I proposed to apply the first (hydrostatic) method to the solution of an equation of the form

$$a_0 + a_1 x + a_2 x^2 + \ldots . + a_n x^n = 0 \ldots . (1).$$

*Mr. Fite first called my attention to this minimum value.